A Thesis Presented to

The Faculty of Alfred University

More than Visual:
Mathematical Concepts Within the Artwork of LeWitt and Escher

by

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In Partial Fulfillment of

the requirements for

the Alfred University Honors Program

May 9, 2019

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Abstract

The goal of this thesis is to demonstrate the relationship between mathematics and art. To do so, I have explored the work of two artists, M.C. Escher and Sol LeWitt. Though these artists approached the role of mathematics in their art in different ways, I have observed that each has employed mathematical concepts in order to create their rule-based artworks. The mathematical ideas which serve as the backbone of this thesis are illustrated by the artists' works and strengthen the bond between the two subjects of art and math. My intention is to make these concepts accessible to all readers, regardless of their mathematical or artistic background, so that they may in turn gain a deeper understanding of the relationship between mathematics and art. To do so, we begin with a philosophical discussion of art and mathematics. Next, we will dissect and analyze various pieces of work by Sol LeWitt and M.C. Escher. As part of that process, we will also redesign or re-imagine some artistic pieces to further highlight mathematical concepts at play within the work of these artists.

1 Introduction

What is art? The Merriam-Webster dictionary provides one definition of art as being "the conscious use of skill and creative imagination especially in the production of aesthetic object" ([1]). However, art is not able to be defined in such a convenient way. Noël Carroll introduces *Theories of Art Today* by stating that a definition of art is nearly impossible [2]. Carroll points to several philosophers who have attempted to create a definition of art by focusing on the few relevant conditions of art and similarities between works already categorized as art and works under consideration of being called "art". However, these philosophers are unable to reach a consensus of what art truly is. Morris Weitz, having attempted the task of creating a working definition of art in 1956, wrote that "the very expansive, adventurous character of art, its ever-present changes and novel creations make it logically impossible to ensure any set of defining properties" ([2], pp. 6).

I believe that the task of defining mathematics falls under the same level of difficulty as the task of defining art. The Merriam-Webster dictionary gives mathematics the technical definition of being "the science of numbers and their operations, interrelations, combinations, generalizations, and abstractions and of space configurations and their structure, measurement, transformations, and generalizations" ([3]). However, much like art, mathematics can not truly or easily be defined due to the fact that it, too, has a "very expansive, adventurous character" with "ever-present changes and novel creations" which "make it logically impossible to ensure any set of defining properties" ([2], pp. 6). Mathematics, like art, consists of a vast amount of topics which are constantly evolving through the discoveries of new connections and completed theorems. Creating a definition in which every single one of these topics may fit is a near impossible task.

It is interesting to note that the various definitions provided in [2] draw parallels to mathematics through the way in which these definitions are constructed. Consider the following definition of a work of art, provided by Marcia Muelder Eaton: "x is a work of art if and only if

1. x is an artifact and

2. x is treated in aesthetically relevant ways; that is, x is treated in such a way that someone who is fluent in a culture is led to direct attention to intrinsic properties of x considered worthy of attention (perception and/or reflection) within that culture and

3. when someone has an aesthetic experience of x, he or she realizes that the

cause of the experience is an intrinsic property of x considered worthy of attention within the culture" ([2], pp. 146).

This use of "if and only if" acts as an equality, meaning that if the first part of the statement is true, then the second part follows and if the second part of the statement is true, then the first part follows. Mathematically, these statements are called biconditional statements, since both parts of the "if and only if" statement must be true in order for the entire claim to be true. A mathematical example of a biconditional statement is "n is an even integer if and only if n^2 is an even integer". In order for a mathematician to prove that this statement is true, they would need to show that "if n is an even integer, then n^2 is an even integer" and "if n^2 is an even integer, then n is an even integer." Thus, Eaton's definition of a work of art has mathematical and logical ties in the sense that it is a biconditional statement and thus, both sides of the statement must be true in order for a piece to be called a work of art.

Another definition of art presented in [2] is Arthur C. Danto's definition that a work of art (1) is always about something, thus having content and meaning and (2) contains something which embodies that meaning ([2], pp. 132). Danto goes on to state his response to the question "What about a painting about nothing?" saying that he "would want to know if it had geometrical forms, nongeometrical forms, whether it was monochrome or striped or what – from this information it is a simple matter to imagine what the appropriate art criticism would be, and to elicit the kind of meaning the work could have" ([2], pp. 132 - 133). Here Danto shows that he would think about pieces mathematically, even before being able to classify them as art using his definition. By considering the geometrical forms or nongeometrical forms present in a work of art, Danto applies a mathematical lens to his viewing of these works, just as I will do throughout my discussion of the works of M.C. Escher and Sol LeWitt. Another interesting point to come out of Danto's definition of art is the question he raises regarding art about nothing. Can works of art truly be about nothing? Certainly not, as all art has been created with some sort of purpose by the artist. Though a work may appear to be meaningless to some viewers, at its core, each piece was built with purpose and at the very least utilizes the basics of the art form it is considered to fall under. A similar phenomenon occurs in mathematics. Though some may consider certain topics of math to be about nothing, all of mathematics are built on axioms at a base level. Thus, these mathematical concepts come from somewhere, showing that their creation was full of intent and purpose, just as the "paintings about nothing" which Danto considers. Therefore, this definition of art provided by Danto has ties to mathematics.

Robert Stecker echoes Noël Carroll's belief that a definition of art is nearly impossible. Stecker's argument branches from the inductive argument, stating that an inclusive definition of art is not possible because "this attempt to define art has failed, that attempt has failed, . . ., so the next attempt will probably fail" ([2], pp. 54). Though a fairly pessimistic view, Stecker's use of induction here is used frequently in mathematics in order to prove that a statement is true for all values $n = k+1$ based on the fact that the statement is true for a starting number $n = k_0$ up to some $n = k$. However, unlike mathematical proofs of induction, Stecker's use of induction in his definition is flawed. In order to correctly utilize induction, the $k+1$ case would need to be proved, instead of solely allowing the assumption that "the next attempt will probably fail." Because Stecker does not provide this proof, his definition of art gives hope for a better, inclusive definition after all. Thus, we are once again provided with a definition of art related to mathematics. As stated, this definition also appears to come closer to an inclusive definition of art, at the very least bringing hope that such a definition may be accomplished. Perhaps considering the relationship between mathematics and art provides the key to completing convenient and inclusive definitions of the two subjects.

The relationship between mathematics and art runs deeper than the similarities in the attempted creation or discovery of their definitions. However, a more in depth relationship between the two subjects is commonly thought to be nonexistent, perhaps due to the different functions of the brain. Most people seem to be aware of the theory that the right side of the brain, which controls the left side of the body, is the more artistic and creative side of the brain. The right side of the brain controls functions such as creativity, art and music awareness, emotions, and imagination. Alternatively, the left side of the brain controls the right side of the body as well as the linguistic and logical functions of the brain. The left side includes functions such as analytic thought, logic, language, math and science, and reasoning ([4], pp. 8 - 10). Because the capabilities of art and math may be physically split into two different hemispheres of the brain, it follows that many people may believe these two subjects to have no correlation with one another.

In my own experience, mathematics also tends to be viewed negatively. When I tell others about my mathematical background, the response tends to either be "Ew, I hate math" accompanied by an eye roll, or high praise because "Good for you. I could never do that. I'm terrible at math!" The arts, on the other hand, receive high praise, as people are drawn to the aesthetics and beauty of great works. In [5], Michele Emmer quotes François Le Lionnais to identify the idea that "In mathematics there exists a beauty which must not be confused with the possible influence of mathematics on the beauty of the works of art. The aesthetics of mathematics must be clearly distinguished from the applications of mathematics to aesthetics ... beauty shows itself in mathematics just as it shows itself in the other sciences, in the arts, in life and in nature" ([5], pp. 353). Though there are many artists who utilize the beauty of mathematics in their work, the focus of this thesis is to discuss the mathematical concepts used by both Maurits Cornelis (M.C.) Escher and Sol LeWitt in their respective rule-based artwork. It is important to note that while Escher and LeWitt both utilized mathematical concepts throughout their works, they approached these concepts in different ways. Though lacking a formal mathematical education, Escher embraced mathematics and overcame his ignorance of the subject by communicating with mathematicians and other scientists. Meanwhile, LeWitt denied any relationship between mathematics and his art, despite the incredibly structured methods with which he created his works.

Born in the Netherlands in 1898, M.C. Escher was a graphic artist who contributed brilliant images of complex concepts to the field of mathematics. Though not a gifted student, Escher embraced the rules and ideas of mathematical concepts in his work, and even communicated with mathematicians and scientists about the accuracy of his pieces. In fact, many of these academics went on to use Escher's art in their own work to illustrate the concepts he was so clearly and beautifully communicating. Reflecting on this phenomenon, Escher once stated,

> "I never got a pass mark in math. The funny thing is I seem to latch on to mathematical theories without realizing what is happening. No indeed, I was a pretty poor pupil at school. And just imagine - mathematicians now use my prints to illustrate their books. Fancy me consorting with all these learned folk, unaware of the fact that I'm ignorant about the whole thing" ([6], pp. 46).

Today, Escher continues to be well known throughout the mathematics community, embraced for his thorough work in exploring tessellations of the plane and hyperbolic geometry. In following the rules of these concepts, Escher has been able to create structured works which stand the test of time.

Unlike Escher, Sol LeWitt viewed his art work as having nothing to do with mathematics whatsoever. Born in 1928, LeWitt was an American artist who contributed greatly to the branch of conceptual art. Despite creating written instructions to construct his structured pieces, LeWitt stated that “Conceptual art doesn’t really have much to do with mathematics . . . The mathematics used by most artists is simple arithmetic or simple number systems” ([7], pp. 848). However, I believe that LeWitt’s work is more related to mathematics than he thought. LeWitt provided written instructions for each of his works, which teams of people followed in order to construct wall drawings comprised of lines and shapes, all aligned in LeWitt’s specific way. This is similar to the way in which mathematicians work with proofs, following logical methods to construct their own proofs as well as using proofs written by others to better understand complex problems. LeWitt’s work with both straight and not straight lines goes further than “simple arithmetic or simple number systems,” and some of his pieces may be considered to be tessellations of the plane or geometrical constructions. LeWitt’s work with color is methodical, as is his work to create his written instructions and masterpieces.

Throughout this thesis, I will provide background on mathematical concepts which I will then supplement with examples from the artists’ collection of works. Using my own mathematical insight, I will provide definitions of the concepts I will be using in order to show that Escher and LeWitt individually applied these same concepts to their art. The more complex ideas presented will further be supplemented with illustrations to fully explain the mathematics within the pieces. I will also address the ways in which each artist used rules to create their structured work, with Escher applying the rules of the mathematical concepts and LeWitt creating written instructions for the creation of his pieces. By implementing these steps, I will show that both M.C. Escher and Sol LeWitt used mathematical concepts to create their rule-based art. Subsequently, I will be able to show how mathematics and art are more closely connected than they may appear to be.

2 More than Visual: Mathematical Concepts Within the Artwork of LeWitt and Escher

M. C. Escher (1898 – 1972) was a Dutch graphic artist who greatly contributed to the mathematical world through his many works of art. Though not a successful student during his school days, Escher was able to create brilliant woodcuts, engravings, mezzotints and lithographs which explored amazing mathematical concepts such as plane tiling and hyperbolic tessellations. Inspired by the Alhambra castle in Granada, Spain, Escher also focused much of his work on symmetry and greatly studied the principle of the division of the plane. Escher's frequent communications with mathematicians and scientists introduced him to the mathematical concepts found in his work, and inspired him to explore these concepts further. Berend, Escher's brother, recommended Hungarian mathematician George Pòlya's 1924 article on the seventeen plane groups of plane symmetry, thus inspiring Escher's work with repetitive patterns. Canadian mathematician H.S.M Coxeter introduced Escher to the idea of hyperbolic tessellations, leading to the artist's series of circle limit work [8].

While M.C. Escher embraced the relationship between mathematics and his artwork, Sol LeWitt believed his art was entirely unrelated to mathematics, stating that "Conceptual art doesn't really have much to do with mathematics . . . The mathematics used by most artists is simple arithmetic or simple number systems" ([7], pp. 848). LeWitt (1928 – 2007) was an American conceptual artist who contributed to the movement of conceptual art and provided the definition that "In conceptual art the idea or concept is the most important aspect of the work . . . all of the planning and decisions are made beforehand and the execution is a perfunctory affair. The idea becomes a machine that makes art" ([7], pp. 846). However, based on this definition, it would appear that conceptual art does, in fact, have strong ties to mathematics. Just as LeWitt refers to ideas as being the machine which makes art, the theorems used by a mathematician act as a machine that makes math. Before providing a written proof, mathematicians must decide which course of action to take throughout the process. They must then plan how they will execute the proof, which often involves several sketches to guide them through the process, allow them to wrap their heads around the issue at hand, and ensure they will not hit any dead ends. In both mathematics and conceptual art, the journey to the output is more important than the solution or art piece itself. One can know the answer to a mathematical problem, however it is more important to understand the theory behind the solution and to be able to know what steps are being taken to reach that answer. In LeWitt's work, the process of creating the piece is more important than the drawing itself, supported by the fact that LeWitt provides written instructions for each wall drawing, which are then executed and put into spaces by his assistants and other groups of volunteers. LeWitt further pointed out the importance of the journey in an interview, stating "I think that any part of the art process, from the inception of the idea in the artist's mind to the inception of the idea in the viewer's mind, all parts are important" ([9], pp. 118).

In embracing the connection between mathematics and art, M.C. Escher used mathematical concepts to create his rule-based artwork. Exploring the concepts freely allowed Escher to learn and fully understand the rules behind them. He was able to create his plane tilings by following the rules outlined by tessellations. Though his work with circle limits is aesthetically pleasing and creative, it is important to note that throughout each piece in this series, Escher is adhering to the rules of hyperbolic geometry. The fact that he discussed these concepts, as well as his art, with mathematicians and scientists further supports the idea that Escher wished to stay true to the laws of mathematics at play in his artwork. Though Sol LeWitt felt differently about the relationship between mathematics and his work, he too utilized mathematical concepts and created rule-based artwork. Because he did not explore mathematical concepts in the same way as Escher, LeWitt's rules come primarily from his own specific written instructions for each of his pieces. Teams of people must then follow these instructions exactly in order to create the vision LeWitt had. This idea of completing a process through an exact order is very mathematical in and of itself. Consider a fairly simple example which most people are familiar with-subtracting two real numbers from one another. Suppose you have the equation

$$37 - 21 = 16$$

Note that if you were to reverse the order of the numbers and instead subtract 37 from 21, you would have the new equation

$$21 - 37 = -16$$

It is clear that these are not the same equations, since $-16 \neq 16$. Thus, this is a basic example of the importance of order in mathematics, which echoes the importance of order in LeWitt's instructions. We will argue that mathematical concepts do, in fact, appear in LeWitt's work, and these concepts are more complex than the "simple arithmetic or simple number systems" which he takes into account. Therefore, both Sol LeWitt and M. C. Escher used mathematical concepts to create their rule-based art.

The basic knowledge of geometrical concepts stems from Euclidean geometry. As the most basic geometry, we are very familiar with Euclidean geometry, though it is worth noting that there are other types of geometries as well. Often taught in high school courses, Euclidean geometry is based on Euclid's axioms. As stated in [10], these axioms are:

1. A (unique) straight line may be drawn from any point to any other point.
2. Every limited straight line can be extended indefinitely to a (unique) straight line.
3. A circle may be drawn with any center and any distance.
4. All right angles are equal.
5. If a straight line intersecting two straight lines makes the interior angles on the same side less than two right angles, then the two lines (if extended indefinitely) will meet on that side on which the angles are less than two right angles.

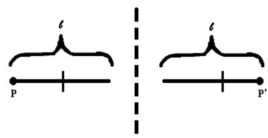
It should be noted that Euclid's fifth postulate is also known as the parallel postulate, which is how it will be referred to here. Unless otherwise noted, the geometrical concepts discussed will be subjects of Euclidean geometry.

A tessellation is defined to be "the tiling of a plane using one or more geometric shapes, called tiles, with no overlaps and no gaps" [11]. Four geometric transformations are crucial to creating tessellations, namely reflection, translation, rotation, and glide reflection. These transformations are also known to mathematicians as isometries, since they preserve both distances and angle measures ([10], pp. 15). Defined mathematically,

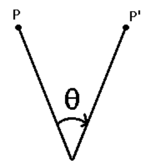
> "A reflection through the line l is an isometry R_l such that it fixes only these points that lie on l and, for each point P not on l, l is the perpendicular bisector of the geodesic segment joining P to $R_l(P)$ ... A rotation about P through the directed angle θ is an isometry

> S_θ that leaves the point P fixed and is such that, for every $Q \neq P$, $S_\theta(Q)$ is on the same circle with center P that Q is on, and the angle $\angle QPS_\theta(Q)$ is congruent to θ and in the same direction ... A translation of distance d along the line l is an isometry T_d that takes each point on l to a point on l at the distance (along l) of d and takes each point not on l to another point on the same side of l and at the same distance from l ... A glide reflection of distance d along the line l is an isometry G_d that takes each point on l to a point on l at the distance (along l) of d and takes each point not on l to another point on the other side of l and at the same distance from l." ([10], pp. 144 – 146).

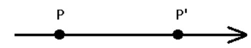
Note that two plane figures are said to be congruent "if, through a combination of translations, rotations, and reflections, one of them can be made to coincide with the other" ([10], pp. 80). A visual representation of these geometric transformations can be seen in figure 1 below.



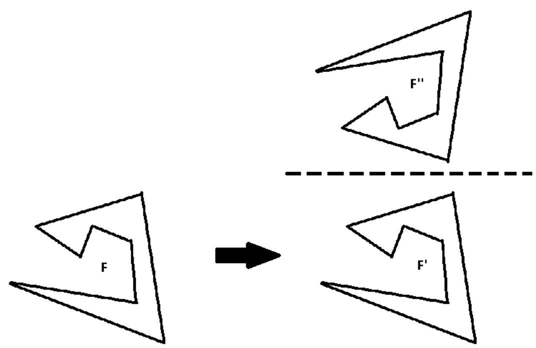
(a) Reflection across the dotted line of point P along line l, such that P' is the reflection of point P



(b) Rotation of point P about angle θ



(c) Translation of point P along line l.



(d) Glide reflection. Note the translation of F to F' in the direction of the arrow, followed by a reflection of F' to F" across the dotted line. See (a) and (c) for a closer view of the two parts of this transformation.

Figure 1: Geometric transformations, illustrated.

How do these geometric transformations allow us to tessellate the plane? To understand this process, we will consider the work of mathematical artist M. C. Escher, who is especially recognized for his work with tessellations. His recognizable patterns of lizards, birds, and fish appear throughout numerous works in which he explores tiling the plane. It is important to note that despite the intricacies and details of Escher's tessellations, each shape has been formed out of a basic geometric polygon. Defined mathematically, a polygon is a closed space made up of n vertices (where $n > 2$), such that the entire shape contains n edges and n angles. A polygon is considered to be regular if all angles and sides are congruent to one another ([10]). Consider, for example, Escher's work *Lizard*, created in 1942 (figure 2a). Figure 2b shows a close up image of one of the lizards used to tessellate the plane.

(a) M.C. Escher, *Lizard*, 1942. India ink, gold ink, colored pencil, poster paint. [12]

(b) Individual lizard tessellation piece

Figure 2: Exploration of Escher's *Lizard*.

Note that the lizard in figure 2b can be shown to have been derived from a geometric polygon, specifically a hexagon. This individual lizard tile may be created through three rotation transformations performed on a regular hexagon. The process is illustrated in figure 3 and outlined in the following steps ([13]):

1. Begin by labeling three non-adjacent vertices of the hexagon as 1, 2, and 3 (red, green, and blue, respectively).

2. Choose one of these points, and alter the side adjacent to it (in this case, altering one side adjacent to vertex 1 to form the first leg of the lizard). Note that it is possible to begin this process with any of the vertices labeled in step 1, as the outcome will result in the same lizard shaped piece.

3. Rotate this altered side about the vertex point, such that it reaches the other adjacent side.

4. Repeat for the other two vertices.

It should be stated that if two figures are adjacent, they are next to each other. Specifically, we will use the term adjacent to refer to figures which share a segment of a side. Thus, two figures are not adjacent if they only share a vertex.

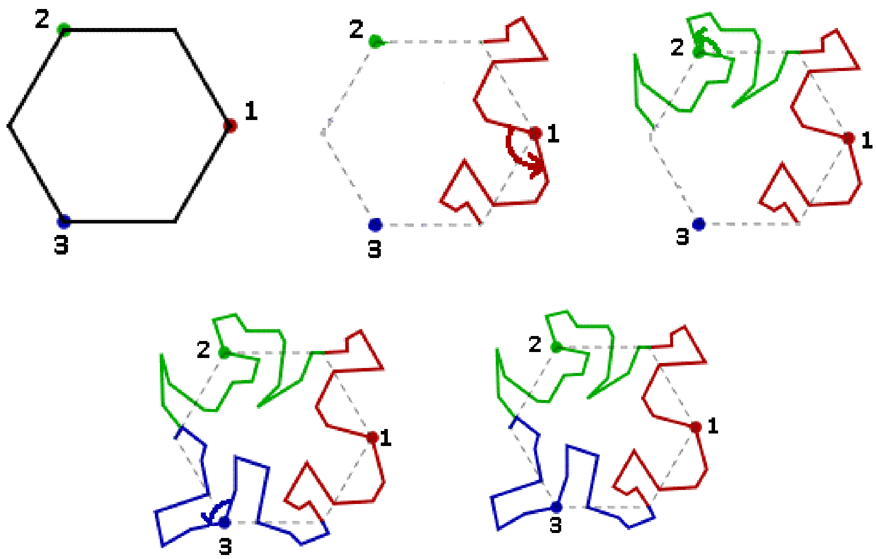


Figure 3: Illustration of creating M.C. Escher's lizard tessellation piece through the rotation transformation ([13]).

The use of geometric transformations to tessellate the plane goes further than creating the tessellation piece itself. In *Lizard* (figure 2a), Escher uses rotation to tessellate the plane with his lizard tiles. Consider a point where the left leg of six lizards meet (figure 4). By rotating the individual lizard tile around this point in a counterclockwise direction, Escher is able to obtain the pattern of six lizards via rotation. It should be noted that while this group of six lizards is obtained from the counterclockwise rotation of the individual lizard tile, the tessellation of the entire plane is obtained by rotating the group of six lizards in a clockwise direction about the point where lizard tails of the same color meet.

Figure 4: Illustration of six lizards, from Escher's *Lizard*, ([12]).

How can we be sure that *Lizard* will result in a complete tessellation of the plane? Geometers know that the only regular polygons which can completely tile a plane are equilateral triangles, squares, and regular hexagons. Recall that for these polygons to be considered regular, they must have congruent side lengths and angles. Other shapes may be able to tile the plane in pairs or groups, however the three polygons mentioned above are the only shapes to create a tessellation on their own (figure 5). Thus, it makes sense that Escher was able to tessellate the plane in *Lizard*, since this piece was constructed by manipulating regular hexagons.

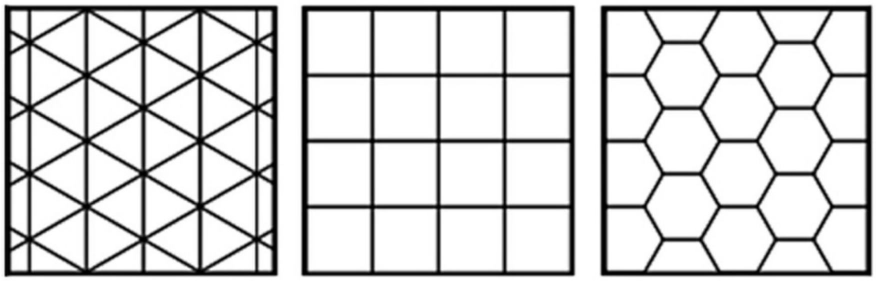
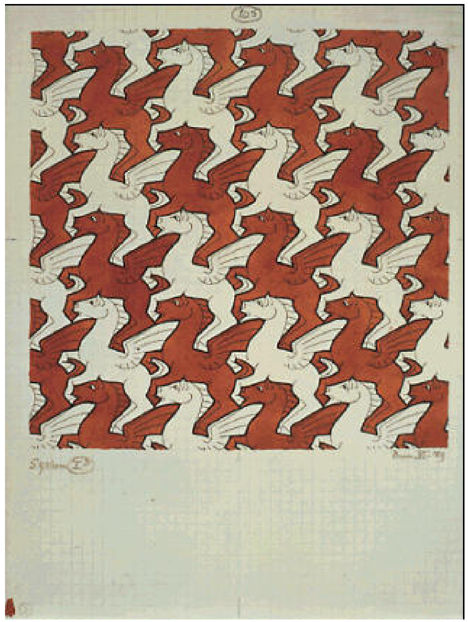


Figure 5: Equilateral triangles, squares and regular hexagons make up regular tessellations. [14]

The process described above of tessellating the plane through rotations is just one way to use geometric transformations to form a tessellation. Following a similar process, one may use reflection, translation, or glide reflection to obtain a tessellation of the plane. Consider Escher's *Pegasus (No. 105)* in figure 6a. Notice that in this print, Escher uses a series of translations to complete the tiling. Though figure 6b only illustrates the translation of the Pegasus tile along the direction of the y-axis, it should be noted that this translation also occurs along the direction of the x-axis of the plane. The more complex method of using glide reflection to complete a tiling of the plane can be seen in Escher's *Swan (No. 96)* (figure 7a). A breakdown of the exact process of completing the glide reflection transformation is outlined in figure 7b. By applying the laws of rotation, reflection, translation, or glide reflection, Escher is able to create his various plane tessellation pieces. Had Escher broken the rules governing the use of these transformations, it is likely he would not have achieved such nice plane tessellations, or otherwise may have been unable to tessellate the plane at all.

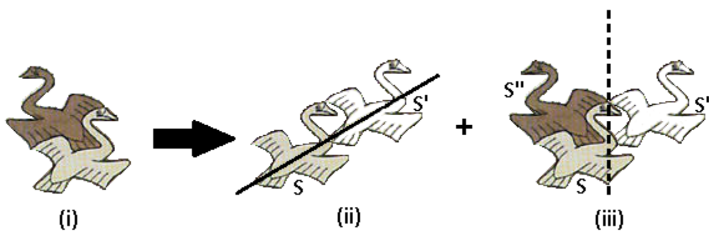


(a) M.C. Escher, *Pegasus (No. 105)*, 1959. India ink, pencil, watercolor. [15].

(b) Closeup image of Escher's *Pegasus (No. 105)*

Figure 6: Exploration of the translation transformation used to tessellate Escher's *Pegasus (No. 105)*

(a) M.C. Escher, *Swan (No. 96)*, 1955.
Ink, watercolor. [16].



(b) Broken down steps of glide reflection used to tessellate Escher's *Swan (No. 96)*. The details of these steps are as follows: (i) Illustrates the final product of the transformation, as appears in the tessellation; (ii) The glide step- S is moved along the line to S' ; (iii) the reflection step- S' is reflected over the dotted line to S" (Note that S is shown in this step as a reference point).

Figure 7: Exploration of the glide reflection transformation used to tessellate Escher's *Swan (No. 96)*

Sol LeWitt also used tessellations in his art, though his basic geometric shaped pieces are not nearly as intricate as Escher's lizards or swans. Consider LeWitt's *Complex Forms* (figure 8). This work fits into our definition of tessellation, since it is made up of one or more geometric shapes which fill the plane without gaps or overlaps. Other works by LeWitt may also be considered as tessellations, provided the viewer is able to extend what they see as a polygon. Recall from before that a polygon is a closed space having n vertices, n edges, and n angles. Consider pieces in which LeWitt covers the plane in bands, for example in *Wall Drawing 419* (figure 10). Though one may see lines when viewing this piece, the instructions call for the construction of bands, suggesting that the piece is made up of two-dimensional quadrilateral shapes (and some five sided shapes, as in the case of some of the diagonally drawn bands) as opposed to one-dimensional lines (figure 9). Thus, the bands used to create *Wall Drawing 419* are polygons, and so with this perspective, the viewer may see that LeWitt is again creating a tessellation of the plane. It is also important to note that the tessellation we are considering in this example is only a snapshot of the plane, since the bands used in *Wall Drawing 419* are truly of infinite length. This is critical to take into account because if one were to consider the bands to be of finite length, then a full tessellation would not be accomplished due to the fact that the plane is infinite. This complicated discussion regarding the bands of *Wall Drawing 419* demonstrates that there are more than just "simple arithmetic and simple number systems" involved in LeWitt's work.

Figure 8: Sol LeWitt, *Complex Forms*, 1990. Color etching. *Artnet.* [17]

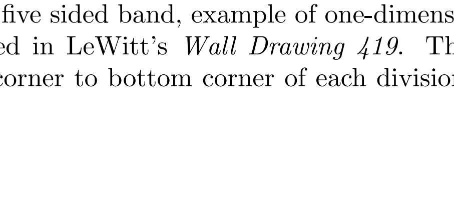


Figure 9: Left to right: Example of two-dimensional quadrilateral band, example of two-dimensional five sided band, example of one-dimensional line. Note that both bands are used in LeWitt's *Wall Drawing 419*. The five sided band is drawn from a top corner to bottom corner of each division in the bottom half of the drawing.

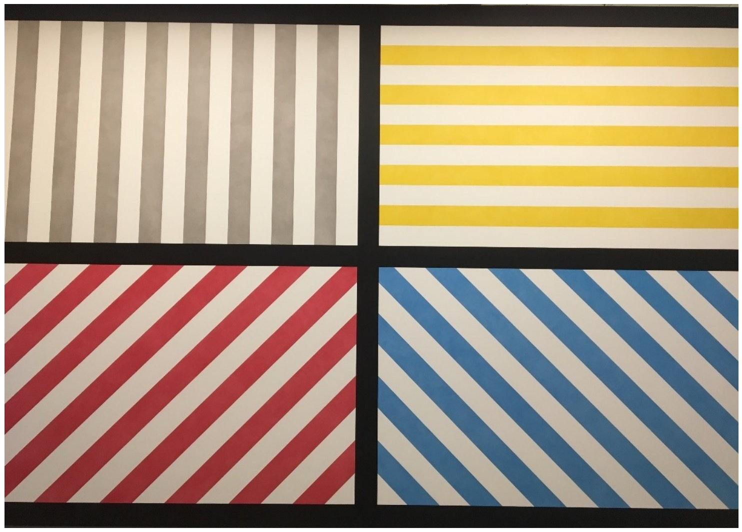
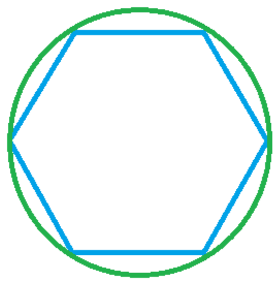


Figure 10: Sol LeWitt, *Wall Drawing 419*, 1984. Color ink wash. Massachusetts Museum of Contemporary Art. [18]

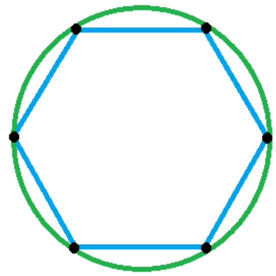
Recall how Escher used a regular hexagon to create his lizard tile used to tessellate the plane in *Lizard*. LeWitt, too, used geometric polygons to derive some of the more complicated geometric forms found in his wall drawings. Consider *Wall Drawing 386* (figure 11). In order to achieve the shape of each star featured in this piece, "a regular polygon is inscribed in a circle. The number of sides the polygon has determines the number of points the star will have. The regularity of the original polygon ensures that the star's points will be evenly spaced. In short, these apparently complex forms are iterations of the basic visual and geometric vocabulary to which LeWitt has committed" ([19]). An example of this process is illustrated in figure 12. Notice that the creation of the star with three points is an exception to this process. The method outlined in figure 12 creates the outline of the full star shape by connecting every individual vertex to all of its non adjacent vertices. Recall that two vertices are considered to be non adjacent if they do not share an edge. However, this cannot work with an equilateral triangle, since each vertex is adjacent to the remaining two.

Thus, a different method must be followed in order for the three pointed star to be constructed. This exception is outlined in figure 14.

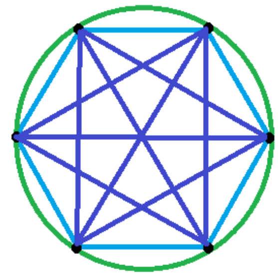
Figure 11: Sol LeWitt, *Wall Drawing 386*, 1983. India ink wash. Massachusetts Museum of Contemporary Art. [20]



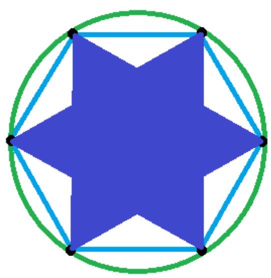
(a) Regular hexagon inscribed in a circle.



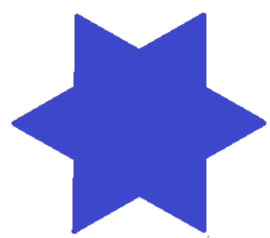
(b) The six points of the hexagon will become the six points of the star. Due to the regularity of the hexagon, these points are evenly spaced.



(c) A line is drawn from each vertex to all of its non adjacent vertices. This provides the outline of the full star shape.

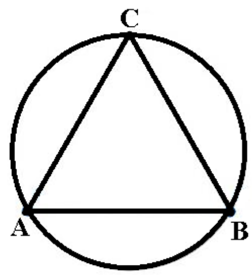


(d) Full six pointed star inscribed in the hexagon within the circle.

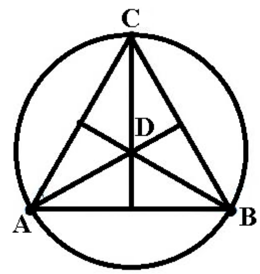


(e) Six pointed star as appears in Sol LeWitt's *wall Drawing 386*.

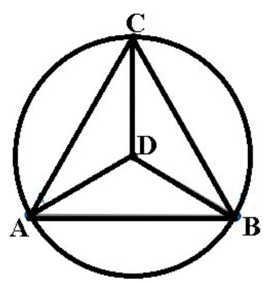
Figure 12: Illustration of creating the six pointed star from LeWitt's *Wall Drawing 386*. With the exception of the three pointed star, all of the other stars in *Wall Drawing 386* are created using this method.



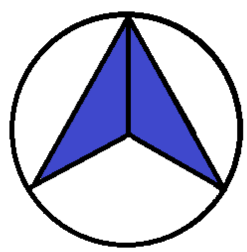
(a) Equilateral triangle inscribed in a circle.



(b) By bisecting each angle, a line is drawn from the vertex to opposite side in order to find the intersection point (also the center point), which has been labeled point D.



(a) This image removes the lines extended past D, such that each segment is only drawn from a vertex point of the triangle to the center point D.



(b) By removing the bottom triangle ($\triangle ADB$ in (c)) and coloring the remaining two triangles, we obtain the three pointed star illustrated in LeWitt's *Wall Drawing 386*.

Figure 14: Illustration of creating the three pointed star from LeWitt's *Wall Drawing 386*. Note that the creation of this star is an exception to the creation of all other stars found in the image.

Pause for a moment to consider the four pointed star in *Wall Drawing 386*, which is derived from a square. Note that the creation of this star presents another exception to the process outlined in figure 12. Though we are able to draw lines from each vertex to all non adjacent vertices (as outlined in figure 12(c)), these lines do not provide the outline of the full star shape as they do for the six pointed star. Instead, the outline produced for the four pointed star must be thickened in order to achieve the star produced by LeWitt in *Wall Drawing 386*. Take some time to consider and explore how this may be done in order to achieve the final product.

Mathematicians know the difficulties of constructing a regular heptagon (seven sided polygon). This difficulty is due to the fact that each interior angle of a heptagon must measure $\frac{900}{7}$ degrees, which cannot be constructed using a compass and straightedge. (A similar difficulty occurs in the construction of the nine sided regular polygon, which requires the construction of a 20° angle, an impossible task to complete using a straightedge and compass alone.) In order to create the regular heptagon necessary for *Wall Drawing 386*, an intern who studies math was brought in to formulate a new construction method. The same draftsman talks about his interest in LeWitt's work as being related to his interest in pure math:

> the same sort of abstract beauty that inspires mathematicians also inspires Sol's work. For instance, the idea of laying out your parameters and exhausting all the possible combinations, and then making so explicit your process, is something that I just find so inspiring and beautiful in the same way that I find pure math ([19]).

This intern's reflection, along with the geometrical method used by LeWitt to construct the stars of *Wall Drawing 386*, demonstrates once again the close ties Sol LeWitt's work has to mathematics. The geometrical constructions of each star in *Wall Drawing 386* proves that LeWitt's piece applies mathematical concepts more complex than the 'simple arithmetic or number systems' which LeWitt claimed were at work. The connection the intern makes between LeWitt's art and pure mathematics further solidifies my belief of a deeper connection between the two subjects. *Wall Drawing 386* is a perfect example of how Sol LeWitt utilizes the beauty of mathematics in his artwork, since he requires the use of complex geometric constructions in order to create the various, aesthetically pleasing stars of the piece.

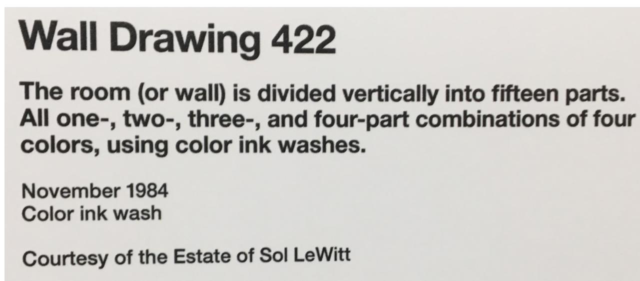


Figure 15: Sol LeWitt's written instructions for *Wall Drawing 422*. See figure 16 for the corresponding artwork.

Apart from the pieces themselves, there are mathematics at work within Sol LeWitt's written instructions. Each list of instructions begins by dividing the wall into various parts, as in figure 15 which begins "The room (or wall) is divided vertically into fifteen parts." Even the colors used in each wall drawing (if the drawing uses colors and is not in black and white) are outlined in the instructions through a type of mathematical recipe. For example, the coloring of *Wall Drawing 422* is described to be "All one-, two-, three-, and four-part combinations of four colors, using color ink washes" (figure 15). Mathematically, this idea of combinations is denoted by the binomial coefficient,

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{when } k \leq n \\ 0 & \text{when } k > n \end{cases}$$

Note that $\binom{n}{k}$ is read as "n choose k", meaning that out of n possibilities, k are chosen. It should also be noted that $n!$ is read as "n factorial" and refers to the multiplication of successive factors. For example, $3! = (3)(2)(1) = 6$. It is also important to know that $0! = 1$. By applying the binomial coefficient, we are also ensuring that each partition of the wall will be colored using a unique color combination, with no repetitions of combinations. Thus, the one-part combinations may be expressed as $\binom{4}{1}$, where of the four colors we are using one of them to color a division of the wall. In other words, the first four divisions of the wall are gray, yellow, red, and blue, respectively. The "two-, three-, and four-part combinations" of these four colors refers to the specific layers of color used on each of the remaining eleven partitions of the wall. Thus, the two-part combinations refers to all possible combinations of two different colors (i.e., $\binom{4}{2}$), the three-part combinations refers to all possible combinations of three different colors (i.e., $\binom{4}{3}$), and the four-part combinations refers to all possible combinations of four different colors (i.e., $\binom{4}{4}$). Further, expressing each of the combinations mathematically in this way allows us to calculate how many divisions of the wall are colored using n-part combinations. Then,

$$\begin{aligned} \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} &= \frac{4!}{1!(4-1)!} + \frac{4!}{2!(4-2)!} + \frac{4!}{3!(4-3)!} + \frac{4!}{4!(4-4)!} \\ &= \frac{4!}{1!(3)!} + \frac{4!}{2!(2)!} + \frac{4!}{3!(1)!} + \frac{4!}{4!(0)!} \\ &= \frac{(4)(3)(2)(1)}{(1)(3)(2)(1)} + \frac{(4)(3)(2)(1)}{(2)(1)(2)(1)} + \frac{(4)(3)(2)(1)}{(3)(2)(1)(1)} + \frac{(4)(3)(2)(1)}{(4)(3)(2)(1)(1)} \\ &= 4 + 6 + 4 + 1 \\ &= 15 \end{aligned}$$

This means that the first four partitions of the wall are colored with one-part combinations, the next six partitions are colored with two-part combinations, three-part combinations color the following four partitions and finally the last partition of the wall is colored by the four-part combination. Thus, the coloring combinations of the entire wall in order are: gray, yellow, red, blue, gray-yellow, gray-red, gray-blue, yellow-red, yellow-blue, red-blue, gray-yellow-red, gray-yellow-blue, gray-red-blue, yellow-red-blue and gray-red-yellow-blue. The fact that the coloring of each division of the wall comes explicitly from LeWitt's written instructions is evidence that there are mathematical concepts at work besides the "simple arithmetic" mentioned by the artist. Though the artist does not state and explain the use of the binomial coefficient as outlined above, mathematicians see that this is an intuitive way to come up with the coloring of each partition of the piece. It should also be noted that the colors could be applied in any order (for example, the three-part combinations could make up the first four divisions of the wall instead of coming after the two-part combinations), however the combinations still come from the binomial coefficient and still determine the number of partitions colored in which combination. For example, $\binom{4}{2} = 6$, thus there are always six partitions colored using a two-part combination, regardless of which six partitions are used. Therefore, this wall drawing is constructed clearly just by the seemingly simple, yet mathematical, instructions provided (figure 16). Further, the mathematics required by both LeWitt, to determine the total number of divisions needed for *Wall Drawing 422*, and the draftsmen, to ensure that all color combinations are achieved, reestablishes LeWitt's comments regarding the importance of the process of creating art.

Figure 16: Sol LeWitt, *Wall Drawing 422*, 1984. Color ink wash. Massachusetts Museum of Contemporary Art. [21].

The discussion regarding Sol LeWitt's *Wall Drawing 422* shows that color follows a specific method in the works of these artists, a method which fits into our mathematical discussion. LeWitt's *Wall Drawing 422* can be mathematically dissected using the binomial coefficient to understand the coloring of each division of the wall. A more in depth discussion on color further supports the idea that this aesthetic feature has deep ties to mathematics.

First proven by Kenneth Appel and Wolfgang Haken in 1976, the Four Color Theorem states that given any separation of a plane into regions, where each region shares a common border with another, no more than four colors are required to color the regions of the plane in such a way that no two adjacent regions have the same color ([22]). Recall that we say two regions are adjacent if they share a segment. Refer back to Escher's *Lizard* (figure 2a). Notice how in this plane tessellation, three colors- red, black, and white- are used for all lizards. Further, no lizard is adjacent to another of the same color, since lizards which share only a vertex are non adjacent, according to our definition. For example, red colored lizards only share full segments with black or white colored lizards. Therefore Escher appears to utilize the Four Color Theorem in this work. It should be noted that Escher appears to apply the Four Color Theorem to most of his works, as may be seen in *Pegasus (No. 105)* (figure

6a) and *Swan (No. 96)* (figure 7a). We will discuss Escher's use of color again shortly in our discussion of *Circle Limit III* (figure 20).

Sol LeWitt uses four colors- notably red, yellow, blue, and gray- again and again to color his pieces. LeWitt's repeated use of primary colors is in and of itself an interesting concept. However, LeWitt does not appear to explicitly use the Four Color Theorem, since his works tend not to minimize the number of colors required to color a piece, as the theorem is meant to do. Consider LeWitt's *Wall Drawing 1112* (figure 17). While LeWitt uses six colors in this wall drawing, no adjacent region has the same color. Perhaps the artist had some version of the theorem in mind when creating this work, or at least appears to have been aware of the ability to color adjacent regions differently with six colors. Either way, as we know from the Four Color Theorem, only four colors are required in order to create this same effect. Since LeWitt's instructions do not explicitly state how many colors must be used for this piece (or even what colors must be used), it is possible to recreate *Wall Drawing 1112* in such a way that it adheres to the Four Color Theorem. Figure 18 shows a re-imagined version of this wall drawing, which I created through an exploration of the Four Color Theorem, using only the colors green, blue, purple, and yellow.

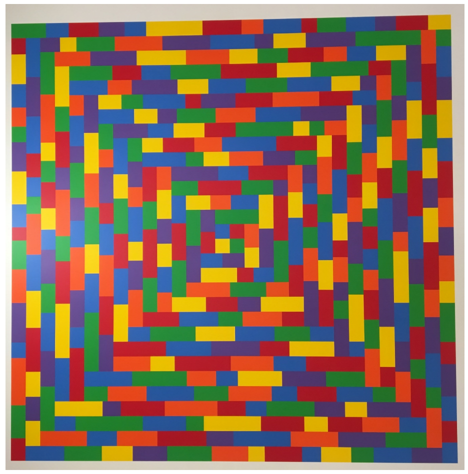


Figure 17: Sol LeWitt, *Wall Drawing 1112*, 2003. Acrylic paint. Massachusetts Museum of Contemporary Art. [23].

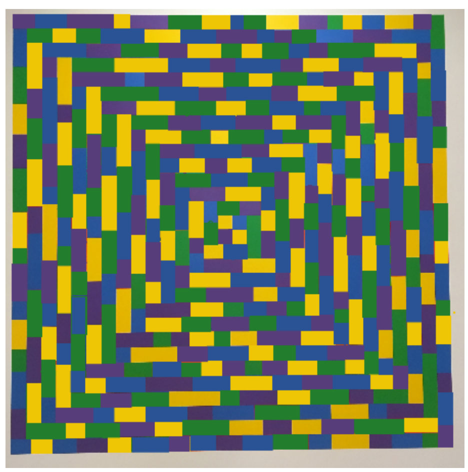


Figure 18: Re-imagined version of LeWitt's *Wall Drawing 1112* which illustrates the Four Color Theorem.

During the exploration of a convenient definition of mathematics, we noted that mathematics as an expansive and ever-growing character. In the spirit of considering this concept, recall that Euclidean geometry is not the only geometry. Mathematicians Janos Bolyai and N.I. Lobachevsky founded a branch of geometry separate from Euclidean geometry, called hyperbolic geometry. Over time, other mathematicians contributed greatly to the concepts of hyperbolic geometry, in both constructing various models and exploring the properties of this geometry. Today, we know hyperbolic geometry to be a geometry which satisfies all postulates of Euclidean geometry except for the parallel postulate. Instead, hyperbolic geometry embraces the concept that there exists more than one straight line through a point in a plane which does not intersect a given line in the plane. In hyperbolic geometry, straight lines also have the property that they may move toward each other without intersecting ([10], pp. 59). In this way, hyperbolic geometry is asymptotic. The French mathematician, Henri Poincaré, greatly contributed to hyperbolic geometry, most notably by providing his Poincaré disk model (figure 19). H.S.M. Coxeter, a Canadian mathematician, used Poincaré's ideas and figures in his writings, which in turn inspired M.C. Escher to create related artwork. In a letter to Coxeter, Escher explains his interest in these ideas by commenting that he is "interested in patterns with 'motives' getting smaller and smaller till they reach the limit of infinite smallness" ([6], pp. 47). This inspiration led Escher to complete his work with circle limits, most notably his piece *Circle Limit III* (figure 20). One can see the relation between the Poincaré disk model and Escher's *Circle Limit III* in the following discussion.

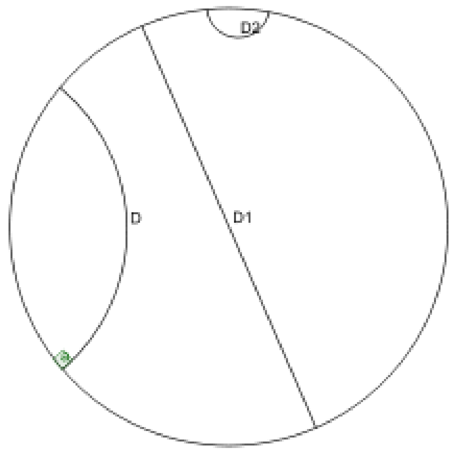


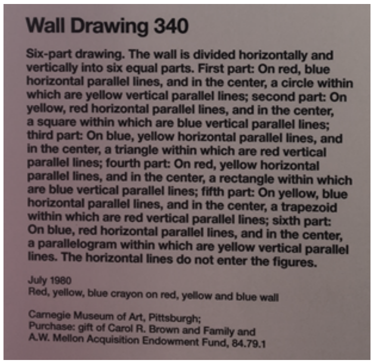
Figure 19: A visual representation of the Poincaré disk model. Because this model uses Euclidean objects to represent hyperbolic geometry, the lines drawn appear to be curved, though they are actually straight in terms of hyperbolic geometry. [24]

Figure 20: M.C. Escher, *Circle Limit III*, 1959. Prints and multiples, woodcut. Escher Museum. [25]

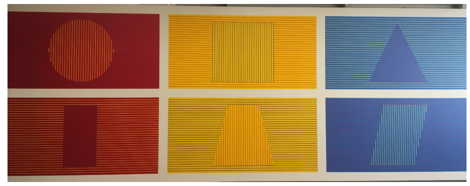
Upon inspection, there are aspects of hyperbolic geometry at work in *Circle Limit III.* Firstly, the pattern of this image is based on the Poincaré disk model of hyperbolic geometry (figure 19). This model uses Euclidean objects to represent objects in hyperbolic geometry ([26], pp. 452). Further, all points in hyperbolic geometry are contained within the interior of the disk, which Escher found to be especially appealing because "an infinitely repeating pattern could be shown in a bounded area and shapes remained recognizable even for small copies of the motif" ([26], pp. 452). It is therefore essential to note that in hyperbolic geometry, distances correspond to ever smaller Euclidean distances toward the edge of the disk. Thus, in the hyperbolic geometry of *Circle Limit III*, each fish in the image is the same size, though they appear to get smaller and smaller in Escher's geometric representation. Escher also utilizes the equidistant curves of hyperbolic geometry, defined to be "curves at a constant hyperbolic distance from the hyperbolic line with the same endpoints on the bounding circle" ([26], pp. 452). Notice these equidistant curves, drawn in white, in figure 20. Escher's use of color in this piece also helps to clearly illustrate these equidistant curves. Notice that fish of the same color not only follow along the same equidistant curve, but follow along the curve in the same direction. Thus, Escher has once again followed the rules of mathematics to create a work of art.

In order to create his pieces tessellating the plane, Escher needed to take into account not only the physical geometric transformations he was using but also the way in which he was allowed to use and create them. As illustrated in figure 3, Escher needed to utilize rotations in a specific way in order to create a lizard piece which could then be used to properly tessellate the plane. This would not be achieved simply by constructing a lizard shape at will and hoping for it to be symmetrical on corresponding sides in order to create the tessellation. Rather, it was necessary for Escher to abide by the rules of tessellations (and subsequently the rules of symmetry and geometric transformations) in order to create these pieces. He would also need to abide by the general rules of Euclidean geometry so that he would even be able to apply the rules of tessellations in the first place. It is also clear that Escher was aware of the mathematical rules he must follow due to his communications with mathematicians and scientists, as well as his studies into their ideas. Similarly, before beginning to create his circle limit prints, Escher needed to be able to follow the laws governing the hyperbolic geometry of these works. He utilized the same rules of tessellations as with his plane tilings in order to tessellate the Poincaré disk model as seen in figure 20. Following the equidistant curves of hyperbolic geometry as well as illustrating the scaling which occurs as a figure moves toward the edge of the disk enabled Escher to create this mesmerizing image while also enabling him to provide a deeper exploration into hyperbolic geometry. In fact, Escher's work allowed mathematicians to illustrate complex mathematical concepts which they otherwise could only imagine. Thus, it is clear that M.C. Escher's artwork is rule-based, due to his necessity to follow the mathematical rules governing his work.

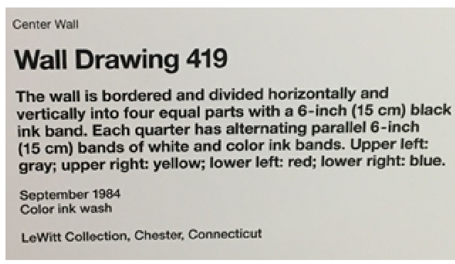
On the other hand, Sol LeWitt created his rule-based artwork by literally creating the rules of each piece. By providing written instructions for each wall drawing, LeWitt was able to lead the groups of people working to construct these pieces in an exact method to create the final product. This is much like the process mathematicians go through when completing a written proof, as they must work from axioms, the rules of logic, and mathematical rules established by others in order to understand the concepts and properly construct their own argument. The instructions themselves are very structured and mathematical, as they consist of precise terms identifying what color combinations to use or exactly what types of lines should be created. Examples of LeWitt's written instructions are seen in figure 21.



(a) Sol LeWitt's written instructions for *Wall Drawing 340*. Corresponding artwork presented in 21b



(b) Sol LeWitt, *Wall Drawing 340*, 1980. Red, yellow, blue crayon on red, yellow and blue wall. Massachusetts Museum of Contemporary Art. [27].



(c) Sol LeWitt's written instructions for *Wall Drawing 419*. See figure 10 for the corresponding artwork.

Figure 21: Examples of the written instructions Sol LeWitt provided in order to create his wall drawings. Taken from the Massachusetts Museum of Contemporary Art.

Although Sol LeWitt's written instructions provide the main basis for his rule-based artwork, the artist adheres to the mathematical rules at play in his pieces as well. Like Escher, Sol LeWitt considers the rules of geometric transformations and symmetry in his pieces which include tessellations. Though not as structured as Escher's tessellations, for example *Lizard* (figure 2a), LeWitt's use of polygons to tile the plane without gaps or overlaps adheres to the basic definition of tessellation, though his initial intended purpose may not have been to tessellate the plane. It is interesting that Sol LeWitt has stated his belief of the disconnection between conceptual art and mathematics, because it appears to be clear that the artist adheres to the basic rules of Euclidean geometry in his works, just as M.C. Escher. Therefore, Sol LeWitt's work does not only follow his own written instructions, but also follows the rules of the mathematical concepts at work.

In fact, I would argue that the execution of LeWitt's instructions is also driven by the rules of mathematics. This claim comes from my experimentation in following LeWitt's instructions to create different versions of the same pieces. Consider the instructions for *Wall Drawing 419* (figure 21c). The instructions for this work specifically dictate which color is to be found in which quadrant of the wall drawing. However, the types of bands used in the drawing are not identified in the same explicit way. Rather, the only instruction provided regarding the creation of the bands is that "each quarter has alternating parallel 6-inch (15 cm) bands of white and color ink bands" (figure 21c). This does not indicate which directions the bands should be drawn or in which specific quadrant of the piece these bands should appear. A sketch of a re-imagined version of *Wall Drawing 419* can be seen in figure 22.

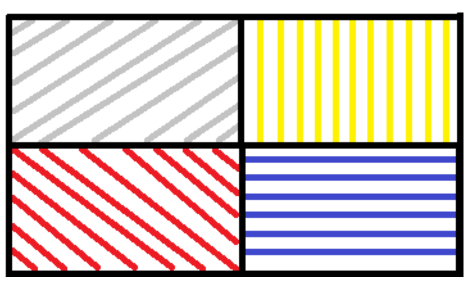


Figure 22: Sketch of re-imagined version of *Wall Drawing 419*.

Note that this re-imagined image adheres to LeWitt's written instructions of the piece, since the coloring and band lengths are as they were meant to be. In fact, the directions of the lines in this re-imagined version use the same four directions as the lines in the original *Wall Drawing 419*. This re-imagined sketch simply mixes up the quadrants the lines are found in. Figure 23 illustrates a more drastic re-imagination of *Wall Drawing 419*, which uses different angles than those used in the original in order to create all of the bands. It is worth mentioning that due to LeWitt's lack of specification regarding the direction of lines in *Wall Drawing 419*, figures 22 and 23 are only two ways variations of the wall drawing. To that regard, there are an infinite number of versions of *Wall Drawing 419*, since there are infinite directions possible for the lines found in the artwork. The wide variety of possibilities of *Wall Drawing 419* speaks to LeWitt's statement that "the idea becomes a machine that makes art" ([7]). In this specific case, the idea refers to LeWitt's *Wall Drawing 419* instructions, which act as the machine to create each and every one of the variations this work can take on. Though the outcomes are all different, they come from the same machine and all follow the same instructions set by LeWitt. Once again, this relates back to mathematics. We discussed earlier how the bands used to create *Wall Drawing 419* are of infinite length. Thus, each part of the piece created from LeWitt's instructions for *Wall Drawing 419* comes from a tessellation of the plane by infinite bands. Therefore, each of the variations of *Wall Drawing 419* are mathematically related, and overall LeWitt's instructions may be captured mathematically. Though there may be one specific way to write a proof, as with the one specific version of Sol LeWitt's written instructions, the idea of the proof may spark different trains of thought with each person who reads it, creating various outcomes stemming from the same initial instructions.

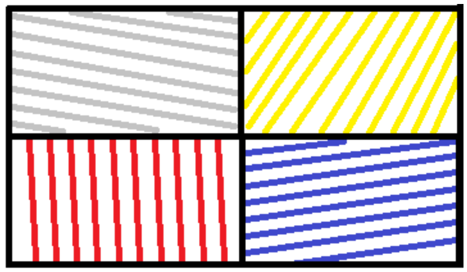


Figure 23: A more drastic sketch of re-imagined version of *Wall Drawing 419*.

Despite M.C. Escher's lack of formal mathematical education and Sol LeWitt's determination that conceptual art is completely unrelated to mathematics, there are many mathematical concepts at work within both artists' creations. Escher's fascination with tiling the plane led to his series on "Regular Divisions of the Plane", all of which adhere to the rules of tessellations and the geometric transformations involved. LeWitt, too, utilizes the rules of tessellations in some of his works, in which he divides the space into sections and uses bands to fill them without overlap or gaps. Escher's communications with other mathematicians and scientists allowed him to create pieces which magnificently illustrated the ideas of hyperbolic geometry - concepts which mathematicians themselves struggled to illustrate. LeWitt's main work with bands and geometric shapes is interesting in both an aesthetic and mathematical sense. Both Sol LeWitt and M.C. Escher have the ability to apply the Four Color Theorem to their pieces, and LeWitt gives further interest to his techniques of coloring by demonstrating mathematical knowledge to achieve the color combinations used to color each division of the wall in his instructions for *Wall Drawing 422*. While Escher's work is fully dictated by the rules of mathematics, LeWitt's instructions provide the specific rules to create each piece. Though these rules do not explicitly outline the steps of the mathematical concepts at work, upon inspection it is clear that there are complex mathematical techniques used by the draftsmen to create each geometric shape or color combination used in LeWitt's work. Thus, though creating art during different time periods and maintaining different views regarding mathematics, both M.C. Escher and Sol LeWitt use mathematical concepts in order to create their rule-based artworks.

Acknowledgments

The author would like to thank her thesis committee advisor, Dr. Amanda Taylor, as well as the other members of her committee, John Hosford and Barbara Lattanzi.

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